

# Lower bounds of eigenvalues of the biharmonic operators by the rectangular Morley element methods \*

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## Abstract

In this paper, we analyze the lower bound property of the discrete eigenvalues by the rectangular Morley elements of the biharmonic operators in both two and three dimensions. The analysis relies on an identity for the errors of eigenvalues. We explore a refined property of the canonical interpolation operators and use it to analyze the key term in this identity. In particular, we show that such a term is of higher order for two dimensions, and is negative and of second order for three dimensions, which causes a main difficulty. To overcome it, we propose a novel decomposition of the first term in the aforementioned identity. Finally, we establish a saturation condition to show that the discrete eigenvalues are smaller than the exact ones. We present some numerical results to demonstrate the theoretical results.

**Keywords:** the rectangular Morley element, the eigenvalue problem, lower bound

## 1 Introduction

We are interested in lower bounds of the eigenvalue problem: Find  $\lambda \in \mathbb{R}$  and  $w \in V := H_0^2(\Omega)$ , such that

$$a(w, v) := (\nabla^2 w, \nabla^2 v)_{L^2(\Omega)} = \lambda(w, v)_{L^2(\Omega)} \quad \text{for any } v \in V. \quad (1.1)$$

where  $\nabla^2 w$  denotes the Hessian matrix of the function  $w$ . In 1979, Rannacher [10] discovered by numerical results that both two dimensional Morley and Adini elements eigenvalues could approximate the exact eigenvalues from below. In 2000, Yang [15] proved such a phenomenon by showing that a consistent error defined there of the Adini element is positive and of second order. The analysis of [15] is based on a key fact that the order of convergence is two for the Adini element eigenfunction in the energy norm. In 2012, Hu and Huang [3] developed a correction operator for the canonical interpolation operators of the Adini element in both two and three dimensions and proved that the discrete eigenvalues are smaller than the exact ones, which employed an identity of the errors of eigenvalues due to [1] and [18], see also [4]. In particular, they showed that the last term in that identity is positive and of second order, which is based on the fact that the order of convergence of the canonical interpolations of the Adini element is two in the energy norm. Besides, Hu, Huang and Lin [4] proposed a new systematic method that

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can produce lower bounds for eigenvalues. In the same paper, they also showed that the Adini element satisfies the condition there and consequently produces lower bounds of eigenvalues. We refer interested readers to [7, 8, 9] for the analysis, based on the expansion method, of the lower bound property of discrete eigenvalues by nonconforming rectangular elements of the Laplace operator in two dimensions. For other related works, we refer to [16] and [17] and the references therein.

The purpose of the current paper is to analyze the lower bound property of the discrete eigenvalues obtained by the rectangular Morley elements. We shall follow [3] and [4] to use the identity from [1, 18]. Note that the elements can not be analyzed by the theory of [4]. In addition, compared with the Adini element analyzed in [15] and [3], the main difficulties for the elements under consideration are

- the order of convergence is only one for both the discrete eigenfunctions and canonical interpolations in the energy norm;
- for the three dimensional element, the last term in the aforementioned identity is negative and of second order.

To overcome these two difficulties, we use the expansion method proposed in [6] to study a refined property of the canonical interpolation operators and propose a novel decomposition of the first term in the identity by using the canonical interpolation operators. Moreover, we prove a saturation condition and employ it to show that the discrete eigenvalues by the two and three dimensional rectangular Morley elements are smaller than the exact ones.

This paper is organized as follows. In the following section, we shall present the two-dimensional rectangular Morley element, and show a refined property of the canonical interpolation operator and use it to prove that the discrete eigenvalues are smaller than the exact ones. In section 3, we present the three-dimensional rectangular Morley element, show a refined property of the canonical interpolation operator, and propose a novel decomposition of the first term in the identity and employ it, after establishing a saturation condition, to prove the lower bound property of discrete eigenvalues. In section 4, we present some numerical results to demonstrate our theoretical results.

## 2 Lower bounds of eigenvalues by the two-dimensional rectangular Morley element

### 2.1 The two-dimensional rectangular Morley element

To consider the discretization of (1.1) by the rectangular Morley element method, let  $\mathcal{T}_h$  be a regular uniform rectangular triangulation of the domain  $\Omega \subset \mathbb{R}^2$  in two dimensions. Given  $K \in \mathcal{T}_h$ , let  $(x_{1,c}, x_{2,c})$  be the center of  $K$ , the meshsize  $h$  and affine mapping:

$$\xi_1 = \frac{x_1 - x_{1,c}}{h}, \quad \xi_2 = \frac{x_2 - x_{2,c}}{h} \quad \text{for any } (x_1, x_2) \in K. \quad (2.1)$$

On element  $K$ , the shape function space of the rectangular Morley element from [11] reads

$$P_T(K) := P_2(K) + \text{span}\{x_1^3, x_2^3\}, \quad (2.2)$$

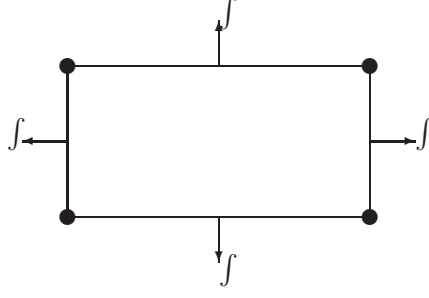


Figure 1: degrees of freedom

here and throughout this paper,  $P_l(K)$  denotes the space of polynomials of degree  $\leq l$  over  $K$ . The nodal parameters are: for any  $v \in C^1(K)$ ,

$$D_T(v) = \left( v(a_i), \frac{1}{|F_j|} \int_{F_j} \frac{\partial v}{\partial \nu_{F_j}} ds \right), \quad i, j = 1, 2, 3, 4, \quad (2.3)$$

where  $a_i$  are vertices of  $K$  and  $F_j$  are edges of  $K$ .  $|F_j|$  denote measure of edges  $F_j$ , see Figure 1. Corresponding to the nodal parameters, the basis functions are the same as those which can be found in [11]. The  $P_T$ -unisolvence of  $D_T$  can be proved similarly as [11]. The nonconforming rectangular Morley element space is then defined by

$$V_h : = \{v \in L^2(\Omega) : v|_K \in P_T(K), \forall K \in \mathcal{T}_h, v \text{ is continuous at all internal vertices and vanishes at all boundary vertices, and } \int_F \frac{\partial v}{\partial \nu_F} ds \text{ is continuous at internal edges } F \text{ and vanishes at boundary edges of } \mathcal{T}_h\}.$$

The finite element approximation of Problem (1.1) reads: Find  $\lambda_h \in \mathbb{R}$  and  $w_h \in V_h$ , such that

$$a_h(w_h, v_h) := (\nabla_h^2 w_h, \nabla_h^2 v_h)_{L^2(\Omega)} = \lambda_h (w_h, v_h)_{L^2(\Omega)} \quad \text{for any } v_h \in V_h. \quad (2.4)$$

where the operator  $\nabla_h^2$  is the discrete counterpart of  $\nabla^2$ , which is defined element by element since the discrete space  $V_h$  is nonconforming.

## 2.2 Interpolation operators

Given  $K \in \mathcal{T}_h$ , define the interpolation operator  $\Pi_K : H^3(K) \rightarrow P_T(K)$  by, for any  $v \in H^3(K)$ ,

$$(\Pi_K v)(P) = v(P) \text{ and } \int_F \frac{\partial \Pi_K v}{\partial \nu_F} ds = \int_F \frac{\partial v}{\partial \nu_F} ds, \quad (2.5)$$

for any vertex  $P$  of  $K$  and any edge  $F$  of  $K$  in the two-dimensional case or any face  $F$  of  $K$  in the three-dimensional case. The interpolation operator  $\Pi_K$  has the following error estimates:

$$|v - \Pi_K v|_{H^l(K)} \leq C h^{3-l} |v|_{H^3(K)}, \quad l = 0, 1, 2, 3, \quad (2.6)$$

provided that  $v \in H^3(K)$ . Herein and throughout this paper,  $C$  denotes a generic positive

constant which is independent of the meshsize and may be different at different places. Then the global version  $\Pi_h$  of the interpolation operator  $\Pi_K$  is defined as

$$\Pi_h|_K = \Pi_K \text{ for any } K \in \mathcal{T}_h. \quad (2.7)$$

In the following, let  $\nabla^l v$  denote the  $l$ -th order tensor of all the  $l$ -th order derivatives of  $v$  and  $\nabla_h^l$  denote the piecewise counterpart of  $\nabla^l$  defined element by element.

Given any element  $K$ , we follow [5] to define  $P_K v \in P_4(K)$  by

$$\int_K \nabla^l P_K v \, dx_1 dx_2 = \int_K \nabla^l v \, dx_1 dx_2, \quad l = 0, 1, 2, 3, 4, \quad (2.8)$$

for two dimensions, and

$$\int_K \nabla^l P_K v \, dx_1 dx_2 dx_3 = \int_K \nabla^l v \, dx_1 dx_2 dx_3, \quad l = 0, 1, 2, 3, 4, \quad (2.9)$$

for three dimensions, for any  $v \in H^4(K)$ . Note that the operator  $P_K$  is well-defined. The interpolation operator  $P_K$  has the following error estimates:

$$\begin{aligned} |v - P_K v|_{H^j(K)} &\leq Ch^{4-j} |v|_{H^4(K)}, \quad j = 0, 1, 2, 3, 4, \\ |v - P_K v|_{H^j(K)} &\leq Ch |v|_{H^{j+1}(K)}, \quad j = 0, 1, 2, 3, \end{aligned} \quad (2.10)$$

provided that  $v \in H^4(K)$ . It follows from the definition of  $P_K$  in (2.9) that

$$\nabla^4 P_K v = \Pi_{0,K} \nabla^4 v, \quad (2.11)$$

with  $\Pi_{0,K}$  the  $L^2$  constant projection operator over  $K$ . The global version  $\Pi_0$  of the interpolation operator  $\Pi_{0,K}$  is defined as

$$\Pi_0|_K = \Pi_{0,K} \text{ for any } K \in \mathcal{T}_h. \quad (2.12)$$

### 2.3 Lower bounds of eigenvalues

In this section, we are in the position to show that the approximate eigenvalues are smaller than the exact ones. Define a semi-norm over  $V_h$  by

$$|u_h|_h^2 := a_h(u_h, u_h) \quad \text{for any } u_h \in V_h.$$

By the error estimates of the interpolation  $\Pi_h u$  and the finite element solution  $u_h$  of the eigenfunction  $u$ , it follows from [11] and [4, 5] that

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^2, \quad |u - u_h|_h^2 \leq Ch^2. \quad (2.13)$$

By the triangle inequality and (2.6), (2.13), we can get that

$$\|\Pi_h u - u_h\|_{L^2(\Omega)} \leq \|u - u_h\|_{L^2(\Omega)} + \|u - \Pi_h u\|_{L^2(\Omega)} \leq Ch^2. \quad (2.14)$$

**Theorem 2.1.** *Let  $(\lambda, u)$  and  $(\lambda_h, u_h)$  be the solutions to (1.1) and (2.4), respectively, and assume that  $u \in H_0^2(\Omega) \cap H^4(\Omega)$ , then,*

$$\lambda_h \leq \lambda,$$

*provided that  $h$  is small enough.*

*Proof.* We need to use an identity for the errors of the eigenvalues from [1, 18], see also [4].

$$\begin{aligned}\lambda - \lambda_h &= |u - u_h|_h^2 - \lambda_h(\Pi_h u - u_h, \Pi_h u - u_h)_{L^2(\Omega)} \\ &\quad + \lambda_h(|\Pi_h u|_{L^2(\Omega)}^2 - \|u\|_{L^2(\Omega)}^2) + 2a_h(u - \Pi_h u, u_h).\end{aligned}\tag{2.15}$$

We can bound the second term by (2.14)

$$(\Pi_h u - u_h, \Pi_h u - u_h)_{L^2(\Omega)} \leq Ch^4\tag{2.16}$$

and the third term as

$$\begin{aligned}||\Pi_h u|_{L^2(\Omega)}^2 - \|u\|_{L^2(\Omega)}^2| &= |(\Pi_h u, \Pi_h u)_{L^2(\Omega)} - (u, u)_{L^2(\Omega)}| \\ &= |(\Pi_h u - u, \Pi_h u)_{L^2(\Omega)} + (u, \Pi_h u - u)_{L^2(\Omega)}| \\ &\leq Ch^3|u|_{H^3(\Omega)}(|\Pi_h u|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}) \\ &\leq Ch^3|u|_{H^3(\Omega)}(|\Pi_h u - u|_{L^2(\Omega)} + 2\|u\|_{L^2(\Omega)}) \\ &\leq Ch^3|u|_{H^3(\Omega)}^2.\end{aligned}\tag{2.17}$$

Herein, we provide a new method to deal with the fourth term. A combination of the first term and the fourth term of (2.15) has the following decomposition

$$\begin{aligned}|u - u_h|_h^2 + 2a_h(u - \Pi_h u, u_h) &= (\nabla_h^2(u - \Pi_h u), \nabla_h^2(u - \Pi_h u))_{L^2(\Omega)} \\ &\quad + (\nabla_h^2(\Pi_h u - u_h), \nabla_h^2(\Pi_h u - u_h))_{L^2(\Omega)} \\ &\quad + 2(\nabla_h^2(u - \Pi_h u), \nabla_h^2(\Pi_h u - u_h))_{L^2(\Omega)} \\ &\quad + 2(\nabla_h^2(u - \Pi_h u), \nabla_h^2 u_h)_{L^2(\Omega)} \\ &= \|\nabla_h^2(u - \Pi_h u)\|_{L^2(\Omega)}^2 + \|\nabla_h^2(\Pi_h u - u_h)\|_{L^2(\Omega)}^2 \\ &\quad + 2(\nabla_h^2(u - \Pi_h u), \nabla_h^2 \Pi_h u)_{L^2(\Omega)}.\end{aligned}\tag{2.18}$$

The third term of (2.18) will be analyzed in Lemma 2.3 below, which reads

$$(\nabla_h^2(u - \Pi_h u), \nabla_h^2 \Pi_h u)_{L^2(\Omega)} \leq \alpha_h h^2,\tag{2.19}$$

where  $\lim_{h \rightarrow 0} \alpha_h = 0$ . It follows from [4] that there holds the saturation condition  $h^2 \leq C|u - u_h|_h^2$ . Hence, by (2.13)–(2.19), the sign of  $\lambda - \lambda_h$  is dominated by the first term and the second term of (2.18).  $\square$

## 2.4 A refined property of the interpolation operator

Given  $K \in \mathcal{T}_h$ , for ease of presentation, we define seven bubble functions with respect to the degrees of freedom defined as in (2.3) as follows

$$\begin{aligned}\varphi_{i,j} &= \xi_i^2 \xi_j - \frac{4}{3} \xi_j + \frac{\xi_j^3}{3}, \quad i, j = 1, 2, \quad i \neq j, \\ \psi_i &= (\xi_i^2 - 1)^2, \quad i = 1, 2, \\ p &= \xi_1^2 + \xi_2^2 - \frac{\xi_1^3}{3} - \frac{\xi_2^3}{3} - \frac{1}{3}, \\ q_{i,j} &= \xi_i^3 \xi_j - \xi_i \xi_j, \quad i, j = 1, 2, \quad i \neq j,\end{aligned}\tag{2.20}$$

In fact, it can be checked that

$$\begin{cases} \varphi_{i,j}(a_k) = 0, & i, j = 1, 2, i \neq j, k = 1, 2, 3, 4, \\ \int_{F_k} \frac{\partial \varphi_{i,j}}{\partial \nu_{F_k}} ds = 0, & i, j = 1, 2, i \neq j, k = 1, 2, 3, 4, \\ \psi_i(a_k) = 0, & i = 1, 2, k = 1, 2, 3, 4, \\ \int_{F_k} \frac{\partial \psi_i}{\partial \nu_{F_k}} ds = 0, & i = 1, 2, k = 1, 2, 3, 4, \\ p(a_k) = 0, & k = 1, 2, 3, 4, \\ \int_{F_k} \frac{\partial p}{\partial \nu_{F_k}} ds = 0, & k = 1, 2, 3, 4, \\ q_{i,j}(a_k) = 0, & i, j = 1, 2, i \neq j, k = 1, 2, 3, 4, \\ \int_{F_k} \frac{\partial q_{i,j}}{\partial \nu_{F_k}} ds = 0, & i, j = 1, 2, i \neq j, k = 1, 2, 3, 4, \end{cases}$$

where  $a_k$  are vertices of  $K$ , and  $F_k$  are edges of  $K$ .

In the next lemma, we follow the idea of [5] to analyze a new refined property for the interpolation operator, which is a basis for the analysis of the term  $a_h(u - \Pi_h u, u_h)$ .

**Lemma 2.2.** *Given  $K \in \mathcal{T}_h$ , for any  $u \in P_4(K)$  and  $v \in P_T(K)$ , there holds that*

$$\begin{aligned} (\nabla^2(u - \Pi_K u), \nabla^2 v)_{L^2(K)} &= \frac{h^2}{3} \int_K \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \frac{\partial^3 v}{\partial x_1^3} dx_1 dx_2 \\ &\quad + \frac{h^2}{3} \int_K \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \frac{\partial^3 v}{\partial x_2^3} dx_1 dx_2. \end{aligned}$$

*Proof.* Let  $\xi_1$  and  $\xi_2$  be defined as in (2.1). It follows from the definition of  $P_T(K)$  that

$$\frac{\partial^2 v}{\partial x_i^2} = \overline{\frac{\partial^2 v}{\partial x_i^2}} + h \frac{\partial^3 v}{\partial x_i^3} \xi_i, \quad i = 1, 2, \quad (2.21)$$

$$\frac{\partial^2 v}{\partial x_1 \partial x_2} = \overline{\frac{\partial^2 v}{\partial x_1 \partial x_2}}, \quad (2.22)$$

where  $\overline{f}$  denotes the integral average of  $f$  over  $K$ , namely,

$$\overline{f} = \frac{1}{|K|} \int_K f dx_1 dx_2.$$

Since  $u \in P_4(K)$ , the Taylor expansion and the definition of the operator  $\Pi_K$  yield

$$\begin{aligned} u - \Pi_K u &= \frac{h^3}{2!} \sum_{i \neq j=1}^2 \overline{\frac{\partial^3 u}{\partial x_i^2 \partial x_j}} \varphi_{i,j} + \frac{h^4}{4!} \sum_{i=1}^2 \frac{\partial^4 u}{\partial x_i^4} \psi_i \\ &\quad + \frac{h^4}{2!2!} \frac{\partial^4 u}{\partial x_1^2 \partial x_2^2} p + \frac{h^4}{3!} \sum_{i \neq j=1}^2 \frac{\partial^4 u}{\partial x_i^3 \partial x_j} q_{i,j}, \end{aligned} \quad (2.23)$$

where  $\varphi_{i,j}$ ,  $\psi_{i,j}$ ,  $p$ , and  $q_{i,j}$  are defined as in (2.20). Hence, the second order partial derivative of  $u - \Pi_K u$  with respect to the variable  $x_1$  reads

$$\begin{aligned} \frac{\partial^2(u - \Pi_K u)}{\partial x_1^2} &= \frac{h}{2!} \overline{\frac{\partial^3 u}{\partial x_1^2 \partial x_2}} 2\xi_2 + \frac{h}{2!} \overline{\frac{\partial^3 u}{\partial x_1 \partial x_2^2}} 2\xi_1 \\ &\quad + \frac{h^2}{4!} \overline{\frac{\partial^4 u}{\partial x_1^4}} (12\xi_1^2 - 4) + \frac{h^2}{2!2!} \overline{\frac{\partial^4 u}{\partial x_1^2 \partial x_2^2}} (2\xi_2^2 - \frac{2}{3}) \\ &\quad + \frac{h^2}{3!} \overline{\frac{\partial^4 u}{\partial x_1^3 \partial x_2}} 6\xi_1 \xi_2. \end{aligned} \quad (2.24)$$

A combination of (2.22) and (2.24) plus some elementary calculation gives

$$\begin{aligned} \int_K \frac{\partial^2(u - \Pi_K u)}{\partial x_1^2} \frac{\partial^2 v}{\partial x_1^2} dx_1 dx_2 &= \int_K \left( \frac{h}{2!} \overline{\frac{\partial^3 u}{\partial x_1^2 \partial x_2}} 2\xi_2 + \frac{h}{2!} \overline{\frac{\partial^3 u}{\partial x_1 \partial x_2^2}} 2\xi_1 + \frac{h^2}{4!} \overline{\frac{\partial^4 u}{\partial x_1^4}} (12\xi_1^2 - 4) + \frac{h^2}{2!2!} \overline{\frac{\partial^4 u}{\partial x_1^2 \partial x_2^2}} (2\xi_2^2 - \frac{2}{3}) \right. \\ &\quad \left. + \frac{h^2}{3!} \overline{\frac{\partial^4 u}{\partial x_1^3 \partial x_2}} 6\xi_1 \xi_2 \right) \left( \overline{\frac{\partial^2 v}{\partial x_1^2}} + h \overline{\frac{\partial^3 v}{\partial x_1^3}} \xi_1 \right) dx_1 dx_2. \end{aligned}$$

Since all coefficients like  $\overline{\frac{\partial^3 u}{\partial x_1^2 \partial x_2}}$  and  $\overline{\frac{\partial^4 u}{\partial x_1^4}}$  are constants, we can get that by parity of functions and symmetry of domains:

$$\begin{aligned} \int_K 2h\xi_2 dx_1 dx_2 &= 0, \quad \int_K 2h\xi_1 dx_1 dx_2 = 0, \quad \int_K h^2(12\xi_1^2 - 4) dx_1 dx_2 = 0, \\ \int_K h^2(2\xi_2^2 - \frac{2}{3}) dx_1 dx_2 &= 0, \quad \int_K h^2 6\xi_1 \xi_2 dx_1 dx_2 = 0. \end{aligned}$$

Hence, only one nonzero term is left, which reads

$$\int_K \frac{h}{2!} \overline{\frac{\partial^3 u}{\partial x_1 \partial x_2^2}} 2\xi_1 h \overline{\frac{\partial^3 v}{\partial x_1^3}} \xi_1 dx_1 dx_2 = \frac{h^2}{3} \int_K \overline{\frac{\partial^3 u}{\partial x_1 \partial x_2^2}} \overline{\frac{\partial^3 v}{\partial x_1^3}} dx_1 dx_2.$$

This yields

$$\int_K \frac{\partial^2(u - \Pi_K u)}{\partial x_1^2} \frac{\partial^2 v}{\partial x_1^2} dx_1 dx_2 = \frac{h^2}{3} \int_K \overline{\frac{\partial^3 u}{\partial x_1 \partial x_2^2}} \overline{\frac{\partial^3 v}{\partial x_1^3}} dx_1 dx_2.$$

A similar argument proves

$$\int_K \frac{\partial^2(u - \Pi_K u)}{\partial x_2^2} \frac{\partial^2 v}{\partial x_2^2} dx_1 dx_2 = \frac{h^2}{3} \int_K \overline{\frac{\partial^3 u}{\partial x_1^2 \partial x_2}} \overline{\frac{\partial^3 v}{\partial x_2^3}} dx_1 dx_2.$$

Finally, the second order mixed partial derivative of  $u - \Pi_K u$  is

$$\begin{aligned} \frac{\partial^2(u - \Pi_K u)}{\partial x_1 \partial x_2} &= \frac{h}{2!} \overline{\frac{\partial^3 u}{\partial x_1^2 \partial x_2}} 2\xi_1 + \frac{h}{2!} \overline{\frac{\partial^3 u}{\partial x_1 \partial x_2^2}} 2\xi_2 \\ &\quad + \frac{h^2}{2!2!} \overline{\frac{\partial^4 u}{\partial x_1^2 \partial x_2^2}} 4\xi_1 \xi_2 + \frac{h^2}{3!} \overline{\frac{\partial^4 u}{\partial x_1^3 \partial x_2}} (3\xi_1^2 - 1) \\ &\quad + \frac{h^2}{3!} \overline{\frac{\partial^4 u}{\partial x_1 \partial x_2^3}} (3\xi_2^2 - 1). \end{aligned} \quad (2.25)$$

A similar procedure of the first part of the proof, this and (2.22) lead to

$$\int_K \frac{\partial^2(u - \Pi_K u)}{\partial x_1 \partial x_2} \frac{\partial^2 v}{\partial x_1 \partial x_2} dx_1 dx_2 = 0, \quad (2.26)$$

which completes the proof.  $\square$

Next, we use Lemma 2.2 to analyze the key term in the proof of Theorem 2.1.

**Lemma 2.3.** *Suppose that  $w \in H_0^2(\Omega) \cap H^4(\Omega)$  with  $\Omega \subset \mathbb{R}^2$ . Then,*

$$(\nabla_h^2(w - \Pi_h w), \nabla_h^2 \Pi_h w)_{L^2(\Omega)} \leq \alpha_h h^2, \quad (2.27)$$

where  $\lim_{h \rightarrow 0} \alpha_h = 0$ .

**Remark 2.4.** *For the Adini element, this term is positive and of order  $O(h^2)$ . However, for the rectangular Morley element, we cannot get a similar result as the Adini element, which indicates a difficulty for the analysis herein.*

*Proof.* Given  $K \in \mathcal{T}_h$ , let the interpolation operator  $P_K$  be defined as in (2.8), which leads to the following decomposition

$$\begin{aligned} (\nabla_h^2(w - \Pi_h w), \nabla_h^2 \Pi_h w)_{L^2(\Omega)} &= \sum_{K \in \mathcal{T}_h} (\nabla_h^2(P_K w - \Pi_K P_K w), \nabla_h^2 \Pi_K w)_{L^2(K)} \\ &\quad + \sum_{K \in \mathcal{T}_h} (\nabla_h^2(I - \Pi_K)(I - P_K)w, \nabla_h^2 \Pi_K w)_{L^2(K)} \\ &= I_1 + I_2. \end{aligned} \quad (2.28)$$

We first analyze the first term  $I_1$  on the right-hand side of (2.28). Let  $u = P_K w$  and  $v = \Pi_K w$  in Lemma 2.2. The first term  $I_1$  on the right-hand side of (2.28) can be rewritten as

$$I_1 = \sum_{i \neq j=1}^2 \sum_{K \in \mathcal{T}_h} \frac{h^2}{3} \int_K \frac{\partial^3 P_K w}{\partial x_i \partial x_j^2} \frac{\partial^3 \Pi_K w}{\partial x_i^3} dx_1 dx_2 = I_{1,1} + I_{1,2}.$$

It is straightforward to show that the first term of  $I_1$  can be expressed as

$$\begin{aligned} I_{1,1} &= \sum_{K \in \mathcal{T}_h} \frac{h^2}{3} \int_K \frac{\partial^3 P_K w}{\partial x_1 \partial x_2^2} \frac{\partial^3 \Pi_K w}{\partial x_1^3} dx_1 dx_2 \\ &= \sum_{K \in \mathcal{T}_h} \frac{h^2}{3} \int_K \frac{\partial^3 w}{\partial x_1 \partial x_2^2} \frac{\partial^3 w}{\partial x_1^3} dx_1 dx_2 + \sum_{K \in \mathcal{T}_h} \frac{h^2}{3} \int_K \frac{\partial^3 (P_K - I)w}{\partial x_1 \partial x_2^2} \frac{\partial^3 \Pi_K w}{\partial x_1^3} dx_1 dx_2 \\ &\quad + \sum_{K \in \mathcal{T}_h} \frac{h^2}{3} \int_K \frac{\partial^3 w}{\partial x_1 \partial x_2^2} \frac{\partial^3 (\Pi_K w - w)}{\partial x_1^3} dx_1 dx_2 \\ &= I_{1,1}^{(1)} + I_{1,1}^{(2)} + I_{1,1}^{(3)}. \end{aligned} \quad (2.29)$$

We are in the position to estimate three terms on the right-hand side of (2.29). Integrating by parts twice and using the fact that

$$\left. \frac{\partial^2 w}{\partial x_1 \partial x_2} \frac{\partial^3 w}{\partial x_1^3} \right|_{\partial \Omega} = \left. \frac{\partial^2 w}{\partial x_1 \partial x_2} \frac{\partial^3 w}{\partial x_1^2 \partial x_2} \right|_{\partial \Omega} = 0$$



show that the first term  $I_{1,1}^{(1)}$  on the right-hand side of (2.29) is

$$I_{1,1}^{(1)} = \sum_{K \in \mathcal{T}_h} \frac{h^2}{3} \int_K \frac{\partial^3 w}{\partial x_1 \partial x_2^2} \frac{\partial^3 w}{\partial x_1^3} dx_1 dx_2 = \frac{h^2}{3} \left\| \frac{\partial^3 w}{\partial x_1^2 \partial x_2} \right\|_{L^2(\Omega)}^2.$$

Since  $\left\| \frac{\partial^3 \Pi_K w}{\partial x_1^3} \right\|_{L^2(K)} \leq C|w|_{H^3(K)}$  is bounded, it follows from (2.10) that

$$I_{1,1}^{(2)} = \sum_{K \in \mathcal{T}_h} \frac{h^2}{3} \int_K \frac{\partial^3 (P_K - I)w}{\partial x_1 \partial x_2^2} \frac{\partial^3 \Pi_K w}{\partial x_1^3} dx_1 dx_2 \leq Ch^3 |w|_{H^4(\Omega)}^2.$$

For the Adini element, the third term  $I_{1,1}^{(3)}$  is a higher order term, since its shape function space contains  $P_3(K)$ . Herein, to analyze the third term  $I_{1,1}^{(3)}$ , we need the following expansion, see (2.23), up to a higher order term,

$$(w - \Pi_K w)|_K = \frac{h^3}{2!} \sum_{i \neq j=1}^2 \frac{\overline{\partial^3 w}}{\partial x_i^2 \partial x_j} \varphi_{i,j} + Ch^4.$$

Since  $\frac{\partial^3 \varphi_{1,2}}{\partial x_1^3}|_K = 0$  and  $\frac{\partial^3 \varphi_{2,1}}{\partial x_1^3}|_K = 2h^{-3}$ , it follows that

$$\begin{aligned} I_{1,1}^{(3)} &= \sum_{K \in \mathcal{T}_h} \frac{h^2}{3} \int_K \frac{\partial^3 w}{\partial x_1 \partial x_2^2} \frac{\partial^3 (\Pi_K w - w)}{\partial x_1^3} dx_1 dx_2 \\ &= -\frac{h^2}{3} \sum_{K \in \mathcal{T}_h} \int_K \left( \frac{\partial^3 w}{\partial x_1 \partial x_2^2} \right)^2 dx_1 dx_2 + Ch^3. \end{aligned}$$

A summary of these previous three equations yields

$$\begin{aligned} I_{1,1} &= I_{1,1}^{(1)} + I_{1,1}^{(2)} + I_{1,1}^{(3)} \\ &= \frac{h^2}{3} \left\| \frac{\partial^3 w}{\partial x_1^2 \partial x_2} \right\|_{L^2(\Omega)}^2 - \frac{h^2}{3} \sum_{K \in \mathcal{T}_h} \int_K \left( \frac{\partial^3 w}{\partial x_1 \partial x_2^2} \right)^2 dx_1 dx_2 + Ch^3. \end{aligned}$$

A similar analysis applies to the second term of  $I_1$ , which implies

$$I_{1,2} = \frac{h^2}{3} \left\| \frac{\partial^3 w}{\partial x_1 \partial x_2^2} \right\|_{L^2(\Omega)}^2 - \frac{h^2}{3} \sum_{K \in \mathcal{T}_h} \int_K \left( \frac{\partial^3 w}{\partial x_1^2 \partial x_2} \right)^2 dx_1 dx_2 + Ch^3.$$

This leads to

$$I_1 = I_{1,1} + I_{1,2} = 0 + Ch^3. \quad (2.30)$$

We turn to the second term  $I_2$  on the right-hand side of (2.28), which can be estimated by the error estimates of (2.10) as

$$|I_2| \leq Ch \sum_{K \in \mathcal{T}_h} \|\nabla_h^3 (I - P_K)w\|_{L^2(K)} |w|_{H^3(\Omega)}.$$

The definition of the projection operator  $P_K$  gives

$$\int_K \nabla^3 (I - P_K)w \, dx_1 dx_2 = 0.$$

By the Poincare inequality, and the commuting property of (2.11),

$$\begin{aligned} |I_2| &\leq Ch^2 \sum_{K \in \mathcal{T}_h} \|\nabla_h^4(I - P_K)w\|_{L^2(K)} |w|_{H^3(\Omega)} \\ &\leq Ch^2 \|(I - \Pi_0)\nabla_h^4 w\|_{L^2(\Omega)} |w|_{H^3(\Omega)}. \end{aligned} \quad (2.31)$$

Since the piecewise constant functions are dense in the space  $L^2(\Omega)$ ,

$$\|(I - \Pi_0)\nabla_h^4 w\|_{L^2(\Omega)} \rightarrow 0, \text{ when } h \rightarrow 0. \quad (2.32)$$

A summary of (2.30), (2.31) and (2.32) completes the proof.  $\square$

**Remark 2.5.** Comparing with the Adini element for the fourth order eigenvalue problem [3], the proof herein weakens the regularity condition from  $H^{4+s}$  with  $(0 < s \leq 1)$  to  $H^4$ .

### 3 Lower bounds of eigenvalues by the three-dimensional rectangular Morley element

The section also uses the identity for the errors of eigenvalue from [1, 18], see also [4]. However, for the three dimensional case, the last term on the right hand side of (3.5) is negative and of order  $O(h^2)$ . This causes a main difficulty. To overcome this difficulty, we propose a new decomposition of the first term by using the canonical interpolation operator defined as in (2.5), see more details in (3.8) below.

#### 3.1 The three-dimensional rectangle Morley element

Let  $\mathcal{T}_h$  be a regular uniform rectangular triangulation of the domain  $\Omega \subset \mathbb{R}^3$  in three dimensions. Given  $K \in \mathcal{T}_h$ , let  $(x_{1,c}, x_{2,c}, x_{3,c})$  be the center of  $K$ , the meshsize  $h$  and affine mapping:

$$\xi_1 = \frac{x_1 - x_{1,c}}{h}, \quad \xi_2 = \frac{x_2 - x_{2,c}}{h}, \quad \xi_3 = \frac{x_3 - x_{3,c}}{h} \text{ for any } (x_1, x_2, x_3) \in K. \quad (3.1)$$

On element  $K$ , the shape function space of the rectangular Morley element reads

$$P_T(K) := P_2(K) + \text{span}\{x_1^3, x_2^3, x_3^3, x_1 x_2 x_3\}. \quad (3.2)$$

The nodal parameters are: for any  $v \in C^1(K)$ ,

$$D_T(v) = \left( v(a_i), \frac{1}{|F_j|} \int_{F_j} \frac{\partial v}{\partial \nu_{F_j}} ds \right), \quad i = 1, \dots, 8, \quad j = 1, \dots, 6, \quad (3.3)$$

where  $a_i$  are vertices of  $K$  and  $F_j$  are faces of  $K$ , see Figure 2. The  $P_T$ -unisolvence of  $D_T$  can be found in [11]. The nonconforming rectangular Morley element space is then defined by

$$\begin{aligned} V_h : &= \{v \in L^2(\Omega) : v|_K \in P_T(K), \forall K \in \mathcal{T}_h, v \text{ is continuous at all internal vertices and} \\ &\text{vanishes at all boundary vertices, and } \int_F \frac{\partial v}{\partial \nu_F} ds \text{ is continuous at internal faces} \\ &F \text{ and vanishes at boundary faces of } \mathcal{T}_h\}. \end{aligned}$$

The finite element approximation of Problem (1.1) reads: Find  $\lambda_h \in \mathbb{R}$  and  $w_h \in V_h$ , such that

$$(\nabla_h^2 w_h, \nabla_h^2 v_h)_{L^2(\Omega)} = \lambda_h (w_h, v_h)_{L^2(\Omega)}, \text{ for any } v_h \in V_h. \quad (3.4)$$

We recall that the operator  $\nabla_h^2$  is the discrete counterpart of  $\nabla^2$ , which is defined element by element since the discrete space  $V_h$  is nonconforming.

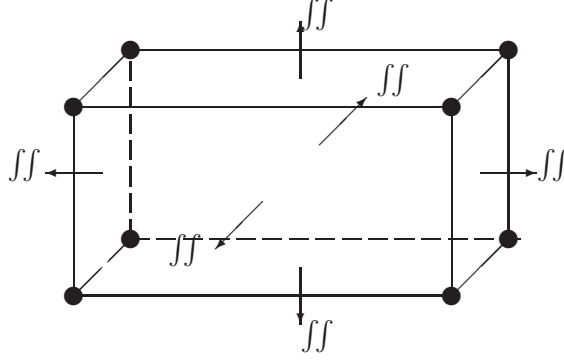


Figure 2: degrees of freedom

### 3.2 Lower bounds of eigenvalues by the three-dimensional rectangular Morley element

In this section, we show that the approximate eigenvalues are smaller than the exact ones in three-dimensional case.

**Theorem 3.1.** *Let  $(\lambda, u)$  and  $(\lambda_h, u_h)$  be the solutions to (1.1) and (3.4), respectively, and assume that  $u \in H_0^2(\Omega) \cap H^4(\Omega)$ , then,*

$$\lambda_h \leq \lambda,$$

*provided that  $h$  is small enough.*

*Proof.* We use the identity for the errors of the eigenvalue from [1, 18], see also [4].

$$\begin{aligned} \lambda - \lambda_h &= |u - u_h|_h^2 - \lambda_h(\Pi_h u - u_h, \Pi_h u - u_h)_{L^2(\Omega)} \\ &\quad + \lambda_h(|\Pi_h u|_{L^2(\Omega)}^2 - |u|_{L^2(\Omega)}^2) + 2a_h(u - \Pi_h u, u_h), \end{aligned} \quad (3.5)$$

We can bound the second term by (2.14)

$$(\Pi_h u - u_h, \Pi_h u - u_h)_{L^2(\Omega)} \leq Ch^4 \quad (3.6)$$

and the third term as

$$\begin{aligned} |||\Pi_h u|_{L^2(\Omega)}^2 - |u|_{L^2(\Omega)}^2| &= |(\Pi_h u, \Pi_h u)_{L^2(\Omega)} - (u, u)_{L^2(\Omega)}| \\ &= |(\Pi_h u - u, \Pi_h u)_{L^2(\Omega)} + (u, \Pi_h u - u)_{L^2(\Omega)}| \\ &\leq Ch^3 |u|_{H^3(\Omega)} (|\Pi_h u|_{L^2(\Omega)} + |u|_{L^2(\Omega)}) \\ &\leq Ch^3 |u|_{H^3(\Omega)} (|\Pi_h u - u|_{L^2(\Omega)} + 2|u|_{L^2(\Omega)}) \\ &\leq Ch^3 |u|_{H^3(\Omega)}^2. \end{aligned} \quad (3.7)$$

Since the fourth term of (3.5) is negative and of order  $O(h^2)$ , we provide a new method to deal with it. A combination of the first term and the fourth term of (3.5) allows for the following decomposition, like (2.18)

$$\begin{aligned} |u - u_h|_h^2 + 2a_h(u - \Pi_h u, u_h) &= \|\nabla_h^2(u - \Pi_h u)\|_{L^2(\Omega)}^2 + \|\nabla_h^2(\Pi_h u - u_h)\|_{L^2(\Omega)}^2 \\ &\quad + 2(\nabla_h^2(u - \Pi_h u), \nabla_h^2 \Pi_h u)_{L^2(\Omega)}. \end{aligned} \quad (3.8)$$

The first term of (3.8) can be further expressed as

$$\begin{aligned}
\|\nabla_h^2(u - \Pi_h u)\|_{L^2(\Omega)}^2 &= \left[ \int_{\Omega} \left( \frac{\partial^2(u - \Pi_h u)}{\partial x_1^2} \right)^2 dx_1 dx_2 dx_3 + \int_{\Omega} \left( \frac{\partial^2(u - \Pi_h u)}{\partial x_2^2} \right)^2 dx_1 dx_2 dx_3 \right. \\
&\quad + \int_{\Omega} \left( \frac{\partial^2(u - \Pi_h u)}{\partial x_3^2} \right)^2 dx_1 dx_2 dx_3 \left. + 2 \left[ \int_{\Omega} \left( \frac{\partial^2(u - \Pi_h u)}{\partial x_1 \partial x_2} \right)^2 dx_1 dx_2 dx_3 \right. \right. \\
&\quad + \int_{\Omega} \left( \frac{\partial^2(u - \Pi_h u)}{\partial x_1 \partial x_3} \right)^2 dx_1 dx_2 dx_3 + \left. \int_{\Omega} \left( \frac{\partial^2(u - \Pi_h u)}{\partial x_2 \partial x_3} \right)^2 dx_1 dx_2 dx_3 \right] \\
&= J_1 + J_2.
\end{aligned}$$

By definition of  $P_T$ , we have

$$\frac{\partial^3 \Pi_h u|_K}{\partial x_i^2 \partial x_j} = 0, \text{ for any } K \in \mathcal{T}_h \text{ and } \Pi_h u \in V_h, \quad i, j = 1, 2, 3, \quad i \neq j. \quad (3.9)$$

Based on this fact, we can show that

$$\beta h^2 \leq J_1 \quad \text{with the constant } \beta > 0. \quad (3.10)$$

See Lemma 3.2 below for a detailed proof of (3.10). To prove the final result, we denote  $J_3 := 2(\nabla_h^2(u - \Pi_h u), \nabla_h^2 \Pi_h u)_{L^2(\Omega)}$ . It will be proved in Lemma 3.5 below that

$$J_2 + J_3 \geq 0 + Ch^3. \quad (3.11)$$

By means of  $J_1, J_2, J_3$ , (3.6)–(3.11), it follows that (3.8) is non-negative and of order  $O(h^2)$ . Therefore, the sign of  $\lambda - \lambda_h$  is non-negative.  $\square$

**Lemma 3.2.** *There holds that*

$$\beta h^2 \leq J_1 \quad \text{with the constant } \beta > 0.$$

*Proof.* Given  $K \in \mathcal{T}_h$ . Let  $P_K$  be defined as in (2.9). By (3.9), it follows from the triangle inequality and the piecewise inverse estimate that

$$\begin{aligned}
\sum_{i \neq j=1}^3 \left\| \frac{\partial^3 u}{\partial x_i^2 \partial x_j} \right\|_{L^2(\Omega)}^2 &= \sum_{i \neq j=1}^3 \sum_{K \in \mathcal{T}_h} \left\| \frac{\partial^3(u - \Pi_h u)}{\partial x_i^2 \partial x_j} \right\|_{L^2(K)}^2 \\
&\leq 2 \sum_{i \neq j=1}^3 \sum_{K \in \mathcal{T}_h} \left( \left\| \frac{\partial^3(u - P_K u)}{\partial x_i^2 \partial x_j} \right\|_{L^2(K)}^2 + \left\| \frac{\partial^3(P_K u - \Pi_h u)}{\partial x_i^2 \partial x_j} \right\|_{L^2(K)}^2 \right) \\
&\leq C \sum_{K \in \mathcal{T}_h} \|\nabla_h^3(u - P_K u)\|_{L^2(K)}^2 + h^{-2} \sum_{i=1}^3 \sum_{K \in \mathcal{T}_h} \left\| \frac{\partial^2(P_K u - \Pi_h u)}{\partial x_i^2} \right\|_{L^2(K)}^2.
\end{aligned}$$

Using the triangle inequality and the error estimate (2.10) leads to

$$\sum_{i \neq j=1}^3 \left\| \frac{\partial^3 u}{\partial x_i^2 \partial x_j} \right\|_{L^2(\Omega)}^2 \leq C \sum_{K \in \mathcal{T}_h} \|\nabla_h^3(u - P_K u)\|_{L^2(K)}^2 + h^{-2} \sum_{i=1}^3 \sum_{K \in \mathcal{T}_h} \left\| \frac{\partial^2(u - \Pi_h u)}{\partial x_i^2} \right\|_{L^2(K)}^2.$$

Using the Poincare inequality and the definition of  $P_K$  yields

$$\sum_{K \in \mathcal{T}_h} \|\nabla_h^3(u - P_K u)\|_{L^2(K)}^2 \leq Ch^2 \|(I - \Pi_0) \nabla_h^4 w\|_{L^2(\Omega)}^2.$$

Since the piecewise constant functions are dense in the space  $L^2(\Omega)$ ,

$$\|(I - \Pi_0)\nabla_h^4 w\|_{L^2(\Omega)} \rightarrow 0 \text{ when } h \rightarrow 0.$$

Finally, it follows that

$$h^2 \sum_{i \neq j=1}^3 \left\| \frac{\partial^3 u}{\partial x_i^2 \partial x_j} \right\|_{L^2(\Omega)}^2 \leq C \sum_{i=1}^3 \sum_{K \in \mathcal{T}_h} \left\| \frac{\partial^2(u - \Pi_h u)}{\partial x_i^2} \right\|_{L^2(K)}^2. \quad (3.12)$$

Next, we prove that  $\sum_{i \neq j=1}^3 \left\| \frac{\partial^3 u}{\partial x_i^2 \partial x_j} \right\|_{L^2(\Omega)}^2 \neq 0$ . In fact, since  $u \in H_0^2(\Omega)$ , if  $\left\| \frac{\partial^3 u}{\partial x_i^2 \partial x_j} \right\|_{L^2(\Omega)} = 0$ , ( $i \neq j = 1, 2, 3$ ),  $u$  should be of the form

$$u(x_1, x_2, x_3) = \sum_{i=1}^3 e_i f_i(x_i) + ax_1 x_2 x_3 + bx_1 x_2 + cx_1 x_3 + dx_2 x_3,$$

for some functions  $f_i(x_i)$  with respect to variable  $x_i$ , and some constants  $e_i, a, b, c, d$ . The boundary condition concludes  $u$  and its normal derivative vanish on the boundary of  $\Omega$ . This implies  $u \equiv 0$ , which contradicts with  $u \neq 0$ .  $\square$

### 3.3 A refined property of the interpolation operator

Given  $K \in \mathcal{T}_h$ , for the sake of simplicity, we define eighteen bubble functions with respect to the degrees of freedom defined as in (3.3) as follows

$$\begin{aligned} \tilde{\varphi}_{i,j} &= \xi_i^2 \xi_j - \frac{4}{3} \xi_j + \frac{\xi_j^3}{3}, \quad i, j = 1, 2, 3, i \neq j, \\ \tilde{\psi}_i &= (\xi_i^2 - 1)^2, \quad i = 1, 2, 3, \\ \tilde{p}_{i,j} &= \xi_i^2 + \xi_j^2 - \frac{\xi_i^3}{3} - \frac{\xi_j^3}{3} - \frac{1}{3}, \quad i = 1, 2, j = 2, 3, i \neq j, \\ \tilde{q}_{i,j} &= \xi_i^3 \xi_j - \xi_i \xi_j, \quad i, j = 1, 2, 3, i \neq j. \end{aligned} \quad (3.13)$$

In fact, it can be checked that

$$\begin{cases} \tilde{\varphi}_{i,j}(a_k) = 0, & i, j = 1, 2, 3, i \neq j, k = 1, \dots, 8, \\ \int_{F_l} \frac{\partial \tilde{\varphi}_{i,j}}{\partial \nu_{F_l}} ds = 0, & i, j = 1, 2, 3, i \neq j, l = 1, \dots, 6, \\ \tilde{\psi}_i(a_k) = 0, & i = 1, 2, 3, k = 1, \dots, 8, \\ \int_{F_l} \frac{\partial \tilde{\psi}_i}{\partial \nu_{F_l}} ds = 0, & i = 1, 2, 3, l = 1, \dots, 6, \\ \tilde{p}_{i,j}(a_k) = 0, & i = 1, 2, j = 2, 3, i \neq j, k = 1, \dots, 8, \\ \int_{F_l} \frac{\partial \tilde{p}_{i,j}}{\partial \nu_{F_l}} ds = 0, & i = 1, 2, j = 2, 3, i \neq j, l = 1, \dots, 6, \\ \tilde{q}_{i,j}(a_k) = 0, & i, j = 1, 2, 3, i \neq j, k = 1, \dots, 8, \\ \int_{F_l} \frac{\partial \tilde{q}_{i,j}}{\partial \nu_{F_l}} ds = 0, & i, j = 1, 2, 3, i \neq j, l = 1, \dots, 6, \end{cases}$$

where  $a_k$  are vertices of  $K$ , and  $F_l$  are faces of  $K$ .

In the next lemma, we follow the idea of [5] to analyze a new refined property for the interpolation operator, which is a basis for the analysis of the term  $a_h(u - \Pi_h u, u_h)$ .

**Lemma 3.3.** *Given  $K \in \mathcal{T}_h$ , for any  $u \in P_4(K)$  and  $v \in P_T(K)$ , there holds that*

$$(\nabla^2(u - \Pi_K u), \nabla^2 v)_{L^2(K)} = \frac{h^2}{3} \sum_{i \neq j=1}^3 \int_K \frac{\partial^3 u}{\partial x_i \partial x_j^2} \frac{\partial^3 v}{\partial x_i^3} dx_1 dx_2 dx_3.$$

*Proof.* Let  $\xi_1, \xi_2$  and  $\xi_3$  be defined as in (3.1). It follows from the definition of  $P_T(K)$  that

$$\begin{aligned}\frac{\partial^2 v}{\partial x_i^2} &= \overline{\frac{\partial^2 v}{\partial x_i^2}} + h \frac{\partial^3 v}{\partial x_i^3} \xi_i, \quad i = 1, 2, 3, \\ \frac{\partial^2 v}{\partial x_i \partial x_j} &= \overline{\frac{\partial^2 v}{\partial x_i \partial x_j}} + h \frac{\partial^3 v}{\partial x_i \partial x_j \partial x_k} \xi_k, \quad i = 1, 2, j = 2, 3, k = 1, 2, 3, i \neq j \neq k.\end{aligned}\tag{3.14}$$

Since  $u \in P_4(K)$ , the Taylor expansion and the definition of the operator  $\Pi_K$  yield

$$\begin{aligned}u - \Pi_K u &= \frac{h^3}{2!} \sum_{i \neq j=1}^3 \overline{\frac{\partial^3 u}{\partial x_i^2 \partial x_j}} \tilde{\varphi}_{i,j} + \frac{h^4}{4!} \sum_{i=1}^3 \frac{\partial^4 u}{\partial x_i^4} \tilde{\psi}_i \\ &\quad + \frac{h^4}{2!2!} \sum_{\substack{i=1,2 \\ j=2,3 \\ i \neq j}} \frac{\partial^4 u}{\partial x_i^2 \partial x_j^2} \tilde{p}_{i,j} + \frac{h^4}{3!} \sum_{i \neq j=1}^3 \frac{\partial^4 u}{\partial x_i^3 \partial x_j} \tilde{q}_{i,j},\end{aligned}\tag{3.15}$$

where,  $\tilde{\varphi}_{i,j}$ ,  $\tilde{\psi}_i$ ,  $\tilde{p}_{i,j}$ , and  $\tilde{q}_{i,j}$  are defined as in (3.13). Hence, the second order partial derivative of  $u - \Pi_K u$  with respect to the variable  $x_i$  reads

$$\begin{aligned}\frac{\partial^2(u - \Pi_K u)}{\partial x_i^2} &= \frac{h}{2!} \sum_{i \neq j=1}^3 \left( \overline{\frac{\partial^3 u}{\partial x_i^2 \partial x_j}} 2\xi_j + \overline{\frac{\partial^3 u}{\partial x_i \partial x_j^2}} 2\xi_i \right) + \frac{h^2}{4!} \frac{\partial^4 u}{\partial x_i^4} (12\xi_i^2 - 4) \\ &\quad + \frac{h^2}{2!2!} \sum_{i \neq j=1}^3 \frac{\partial^4 u}{\partial x_i^2 \partial x_j^2} (2\xi_j^2 - \frac{2}{3}) + \frac{h^2}{3!} \sum_{i \neq j=2}^3 \frac{\partial^4 u}{\partial x_i^3 \partial x_j} 6\xi_i \xi_j,\end{aligned}\tag{3.16}$$

A similar argument for the two-dimensional case shows

$$\int_K \frac{\partial^2(u - \Pi_K u)}{\partial x_i^2} \frac{\partial^2 v}{\partial x_i^2} dx_1 dx_2 dx_3 = \frac{h^2}{3} \sum_{i \neq j=1}^3 \int_K \frac{\partial^3 u}{\partial x_i \partial x_j^2} \frac{\partial^3 v}{\partial x_i^3} dx_1 dx_2 dx_3.$$

The second order mixed partial derivative of  $u - \Pi_K u$  is

$$\begin{aligned}\frac{\partial^2(u - \Pi_K u)}{\partial x_i \partial x_j} &= \frac{h}{2!} \overline{\frac{\partial^3 u}{\partial x_i^2 \partial x_j}} 2\xi_i + \frac{h}{2!} \overline{\frac{\partial^3 u}{\partial x_i \partial x_j^2}} 2\xi_j + \frac{h^2}{2!2!} \frac{\partial^4 u}{\partial x_i^2 \partial x_j^2} 4\xi_i \xi_j + \\ &\quad \frac{h^2}{3!} \frac{\partial^4 u}{\partial x_i^3 \partial x_j} (3\xi_i^2 - 1) + \frac{h^2}{3!} \frac{\partial^4 u}{\partial x_i \partial x_j^3} (3\xi_j^2 - 1), \\ &\quad i = 1, 2, j = 2, 3, i \neq j.\end{aligned}$$

This and (3.14) lead to

$$\int_K \frac{\partial^2(u - \Pi_K u)}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} dx_1 dx_2 dx_3 = 0, \quad i = 1, 2, j = 2, 3, i \neq j,\tag{3.17}$$

which completes the proof.  $\square$

Next, we use Lemma 3.3 to analyze the key term in the proof of Theorem 3.1.

**Lemma 3.4.** *Suppose that  $w \in H_0^2(\Omega) \cap H^4(\Omega)$  with  $\Omega \subset \mathbb{R}^3$ . Then,*

$$\begin{aligned} (\nabla_h^2(w - \Pi_h w), \nabla_h^2 \Pi_h w)_{L^2(\Omega)} &= -\frac{h^2}{3} \sum_{i \neq j \neq k=1}^3 \sum_{K \in \mathcal{T}_h} \int_K \frac{\partial^3 w}{\partial x_i \partial x_j^2} \frac{\partial^3 w}{\partial x_i \partial x_k^2} dx_1 dx_2 dx_3 \\ &\quad + Ch^3. \end{aligned}$$

*Proof.* Given  $K \in \mathcal{T}_h$ , let the interpolation operator  $P_K$  be defined as in (2.9), which leads to the following decomposition

$$\begin{aligned} (\nabla_h^2(w - \Pi_h w), \nabla_h^2 \Pi_h w)_{L^2(\Omega)} &= \sum_{K \in \mathcal{T}_h} (\nabla_h^2(P_K w - \Pi_K P_K w), \nabla_h^2 \Pi_K w)_{L^2(K)} \\ &\quad + \sum_{K \in \mathcal{T}_h} (\nabla_h^2(I - \Pi_K)(I - P_K)w, \nabla_h^2 \Pi_K w)_{L^2(K)} \\ &= I_1 + I_2. \end{aligned} \tag{3.18}$$

Let  $u = P_K w$  and  $v = \Pi_K w$  in Lemma 3.3, the first term  $I_1$  on the right-hand side of (3.18) can be rewritten as

$$I_1 = \frac{h^2}{3} \sum_{i \neq j=1}^3 \sum_{K \in \mathcal{T}_h} \int_K \frac{\partial^3 P_K w}{\partial x_i \partial x_j^2} \frac{\partial^3 \Pi_K w}{\partial x_i^3} dx_1 dx_2 dx_3 = \sum_{i \neq j=1}^3 I_1^{(i,j)}.$$

The term  $I_1^{(i,j)}$  has the following decomposition

$$\begin{aligned} I_1^{(i,j)} &= \sum_{K \in \mathcal{T}_h} \frac{h^2}{3} \int_K \frac{\partial^3 w}{\partial x_i \partial x_j^2} \frac{\partial^3 w}{\partial x_i^3} dx_1 dx_2 dx_3 \\ &\quad + \sum_{K \in \mathcal{T}_h} \frac{h^2}{3} \int_K \frac{\partial^3 w}{\partial x_i \partial x_j^2} \frac{\partial^3 (\Pi_K w - w)}{\partial x_i^3} dx_1 dx_2 dx_3 \\ &\quad + \sum_{K \in \mathcal{T}_h} \frac{h^2}{3} \int_K \frac{\partial^3 (P_K - I)w}{\partial x_i \partial x_j^2} \frac{\partial^3 \Pi_K w}{\partial x_i^3} dx_1 dx_2 dx_3 \\ &= I_{1,1}^{(i,j)} + I_{1,2}^{(i,j)} + I_{1,3}^{(i,j)}. \end{aligned}$$

After integrating by parts twice, the first term of  $I_1^{(i,j)}$  can be expressed as

$$I_{1,1}^{(i,j)} = \sum_{K \in \mathcal{T}_h} \frac{h^2}{3} \int_K \frac{\partial^3 w}{\partial x_i \partial x_j^2} \frac{\partial^3 w}{\partial x_i^3} dx_1 dx_2 dx_3 = \frac{h^2}{3} \left\| \frac{\partial^3 w}{\partial x_i^2 \partial x_j} \right\|_{L^2(\Omega)}^2.$$

Since  $\left\| \frac{\partial^3 \Pi_K w}{\partial x_i^3} \right\|_{L^2(K)} \leq C|w|_{H^3(K)}$  is bounded, it follows that

$$I_{1,3}^{(i,j)} = \sum_{K \in \mathcal{T}_h} \frac{h^2}{3} \int_K \frac{\partial^3 (P_K - I)w}{\partial x_i \partial x_j^2} \frac{\partial^3 \Pi_K w}{\partial x_i^3} dx_1 dx_2 dx_3 \leq Ch^3 |w|_{H^4(\Omega)}^2.$$

Due to the Taylor expansion, see (3.15), up to a higher order term, it follows that

$$\begin{aligned}
I_{1,2}^{(i,j)} &= \sum_{K \in \mathcal{T}_h} \frac{h^2}{3} \int_K \frac{\partial^3 w}{\partial x_i \partial x_j^2} \frac{\partial^3 (\Pi_K w - w)}{\partial x_i^3} dx_1 dx_2 dx_3 \\
&= - \sum_{K \in \mathcal{T}_h} \frac{h^2}{3} \int_K \frac{\partial^3 w}{\partial x_i \partial x_j^2} \frac{\partial^3 w}{\partial x_i \partial x_j^2} dx_1 dx_2 dx_3 - \sum_{K \in \mathcal{T}_h} \frac{h^2}{3} \int_K \frac{\partial^3 w}{\partial x_i \partial x_k^2} \frac{\partial^3 w}{\partial x_i \partial x_j^2} dx_1 dx_2 dx_3 \\
&\quad + Ch^3 \\
&= - \frac{h^2}{3} \sum_{K \in \mathcal{T}_h} \int_K \left( \frac{\partial^3 w}{\partial x_i \partial x_j^2} \right)^2 dx_1 dx_2 dx_3 - \frac{h^2}{3} \sum_{K \in \mathcal{T}_h} \int_K \frac{\partial^3 w}{\partial x_i \partial x_k^2} \frac{\partial^3 w}{\partial x_i \partial x_j^2} dx_1 dx_2 dx_3 \\
&\quad + Ch^3, \quad i \neq j \neq k = 1, 2, 3.
\end{aligned}$$

A summary of these three terms leads to

$$\begin{aligned}
I_1^{(i,j)} &= \frac{h^2}{3} \left\| \frac{\partial^3 w}{\partial x_i^2 \partial x_j} \right\|_{L^2(\Omega)}^2 - \frac{h^2}{3} \sum_{K \in \mathcal{T}_h} \int_K \left( \frac{\partial^3 w}{\partial x_i \partial x_j^2} \right)^2 dx_1 dx_2 dx_3 \\
&\quad - \frac{h^2}{3} \sum_{K \in \mathcal{T}_h} \int_K \frac{\partial^3 w}{\partial x_i \partial x_j^2} \frac{\partial^3 w}{\partial x_i \partial x_k^2} dx_1 dx_2 dx_3 + Ch^3, \quad i \neq j \neq k = 1, 2, 3.
\end{aligned}$$

It is a consequence of the sum of  $I_1^{(i,j)}$  that

$$I_1 = - \frac{h^2}{3} \sum_{i \neq j \neq k=1}^3 \sum_{K \in \mathcal{T}_h} \int_K \frac{\partial^3 w}{\partial x_i \partial x_j^2} \frac{\partial^3 w}{\partial x_i \partial x_k^2} dx_1 dx_2 dx_3 + Ch^3. \quad (3.19)$$

We turn to the second term  $I_2$  on the right-hand side of (3.18), which can be estimated by the error estimates of (2.10) as

$$|I_2| \leq Ch \sum_{K \in \mathcal{T}_h} \|\nabla_h^3 (I - P_K)w\|_{L^2(K)} |w|_{H^3(\Omega)}.$$

The definition of the projection operator  $P_K$  gives

$$\int_K \nabla_h^3 (I - P_K)w \, dx_1 dx_2 dx_3 = 0.$$

By the Poincare inequality, and the commuting property of (2.11),

$$\begin{aligned}
|I_2| &\leq Ch^2 \sum_{K \in \mathcal{T}_h} \|\nabla_h^4 (I - P_K)w\|_{L^2(K)} |w|_{H^3(\Omega)} \\
&\leq Ch^2 \|(I - \Pi_0) \nabla_h^4 w\|_{L^2(\Omega)} |w|_{H^3(\Omega)}.
\end{aligned} \quad (3.20)$$

Since the piecewise constant functions are dense in the space  $L^2(\Omega)$ ,

$$\|(I - \Pi_0) \nabla_h^4 w\|_{L^2(\Omega)} \rightarrow 0 \text{ when } h \rightarrow 0. \quad (3.21)$$

A summary of (3.19), (3.20) and (3.21) completes the proof.  $\square$

**Lemma 3.5.** *Let  $u$  be the eigenfunction of Problem (1.1). Assume that  $u \in H_0^2(\Omega) \cap H^4(\Omega)$  with  $\Omega \subset \mathbb{R}^3$ , then*

$$J_2 + J_3 \geq 0 + Ch^3.$$



*Proof.*  $J_2$  can be expressed as

$$\begin{aligned} J_2 &= 2 \int_{\Omega} \left[ \left( \frac{\partial^2(u - \Pi_h u)}{\partial x_1 \partial x_2} \right)^2 + \left( \frac{\partial^2(u - \Pi_h u)}{\partial x_1 \partial x_3} \right)^2 + \left( \frac{\partial^2(u - \Pi_h u)}{\partial x_2 \partial x_3} \right)^2 \right] dx_1 dx_2 dx_3 \\ &= 2 \sum_{K \in \mathcal{T}_h} \int_K \left[ \left( \frac{\partial^2(u - \Pi_K u)}{\partial x_1 \partial x_2} \right)^2 + \left( \frac{\partial^2(u - \Pi_K u)}{\partial x_1 \partial x_3} \right)^2 + \left( \frac{\partial^2(u - \Pi_K u)}{\partial x_2 \partial x_3} \right)^2 \right] dx_1 dx_2 dx_3. \end{aligned}$$

By the Taylor expansion (3.15), up to a higher order term,  $u - \Pi_K u$  has the following expression,

$$\begin{aligned} u - \Pi_K u &= \frac{h^3}{2!} \sum_{i \neq j=1}^3 \overline{\frac{\partial^3 u}{\partial x_i^2 \partial x_j}} \tilde{\varphi}_{i,j} + O(h^4) \\ &= \frac{h^3}{2!} \sum_{i \neq j=1}^3 \overline{\frac{\partial^3 u}{\partial x_i^2 \partial x_j}} \left( \xi_i^2 \xi_j - \frac{4}{3} \xi_j + \frac{\xi_j^3}{3} \right) + O(h^4), \end{aligned}$$

which  $\xi_i$  are defined as in (3.1). The second order mixed partial derivative of  $u - \Pi_K u$  yields

$$\frac{\partial^2(u - \Pi_K u)}{\partial x_i \partial x_j} = \frac{h}{2!} \overline{\frac{\partial^3 u}{\partial x_i^2 \partial x_j}} 2\xi_i + \frac{h}{2!} \overline{\frac{\partial^3 u}{\partial x_i \partial x_j^2}} 2\xi_j + O(h^2), \quad i = 1, 2, j = 2, 3, i \neq j.$$

Since  $\overline{\frac{\partial^3 u}{\partial x_i^2 \partial x_j}}$  and  $\overline{\frac{\partial^3 u}{\partial x_i \partial x_j^2}}$  are constants, we can get that by parity of functions and symmetry of domains:

$$\int_K \overline{\frac{\partial^3 u}{\partial x_i^2 \partial x_j}} \overline{\frac{\partial^3 u}{\partial x_i \partial x_j^2}} \xi_i \xi_j dx_1 dx_2 dx_3 = 0, \quad i = 1, 2, j = 2, 3, i \neq j.$$

This yields

$$\begin{aligned} \int_K \left( \frac{\partial^2(u - \Pi_K u)}{\partial x_i \partial x_j} \right)^2 dx_1 dx_2 dx_3 &= h^2 \int_K \left( \frac{\partial^3 u}{\partial x_i^2 \partial x_j} \right)^2 \xi_i^2 dx_1 dx_2 dx_3 \\ &\quad + h^2 \int_K \left( \frac{\partial^3 u}{\partial x_i \partial x_j^2} \right)^2 \xi_j^2 dx_1 dx_2 dx_3 + O(h^4), \\ &\quad i = 1, 2, j = 2, 3, i \neq j. \end{aligned}$$

Thus,  $J_2$  can be rewritten as

$$J_2 = 2h^2 \sum_{i \neq j=1}^3 \sum_{K \in \mathcal{T}_h} \int_K \left( \frac{\partial^3 u}{\partial x_i^2 \partial x_j} \right)^2 \xi_i^2 dx_1 dx_2 dx_3 + O(h^4).$$

Concerning the last term of (3.8), in Lemma 3.4, it follows that

$$\begin{aligned} (\nabla_h^2(u - \Pi_h u), \nabla_h^2 \Pi_h u)_{L^2(\Omega)} &= -\frac{h^2}{3} \sum_{i \neq j \neq k=1}^3 \sum_{K \in \mathcal{T}_h} \int_K \frac{\partial^3 u}{\partial x_i \partial x_j^2} \frac{\partial^3 u}{\partial x_i \partial x_k^2} dx_1 dx_2 dx_3 \\ &\quad + Ch^3. \end{aligned} \tag{3.22}$$

By the Cauchy-Schwartz inequality, this yields

$$J_3 \geq -\frac{2h^2}{3} \sum_{i \neq j=1}^3 \sum_{K \in \mathcal{T}_h} \int_K \left( \frac{\partial^3 u}{\partial x_i \partial x_j^2} \right)^2 dx_1 dx_2 dx_3 + Ch^3.$$

Hence,

$$\begin{aligned}
J_2 + J_3 &\geq 2h^2 \sum_{i \neq j=1}^3 \sum_{K \in \mathcal{T}_h} \int_K \left( \frac{\partial^3 u}{\partial x_i^2 \partial x_j} \right)^2 \xi_i^2 dx_1 dx_2 dx_3 \\
&\quad - \frac{2h^2}{3} \sum_{i \neq j=1}^3 \sum_{K \in \mathcal{T}_h} \int_K \left( \frac{\partial^3 u}{\partial x_i \partial x_j^2} \right)^2 dx_1 dx_2 dx_3 + Ch^3.
\end{aligned}$$

Given  $K \in \mathcal{T}_h$ , it is sufficient to analyze the following term

$$\begin{aligned}
&2h^2 \int_K \left( \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right)^2 \xi_1^2 dx_1 dx_2 dx_3 - \frac{2h^2}{3} \int_K \left( \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right)^2 dx_1 dx_2 dx_3 \\
&= 2h^2 \int_K \left( \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right)^2 \left( \xi_1^2 - \frac{1}{3} \right) dx_1 dx_2 dx_3 \\
&= 2h^2 \int_K \left[ \left( \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right)^2 - \left( \overline{\frac{\partial^3 u}{\partial x_1^2 \partial x_2}} \right)^2 \right] \left( \xi_1^2 - \frac{1}{3} \right) dx_1 dx_2 dx_3 \\
&\quad + 2h^2 \int_K \left( \overline{\frac{\partial^3 u}{\partial x_1^2 \partial x_2}} \right)^2 \left( \xi_1^2 - \frac{1}{3} \right) dx_1 dx_2 dx_3 \\
&\leq Ch^3 |u|_{H^4(K)}^2,
\end{aligned}$$

where we use the fact that the second term of the second identity vanishes.  $\square$

## 4 Numerical results

In this section, we present some numerical results to demonstrate our theoretical results. Herein, we denote  $r$  as the rate of convergence. In the first example, we consider the following eigenvalue problem of the two-dimensional biharmonic equation imposed the following boundary conditions

$$\Delta^2 u = \lambda u, \text{ in } \Omega = [0, 1]^2, \quad (4.1)$$

(a) The clamped boundary condition

$$u = \partial_\nu u = 0 \text{ on } \partial\Omega.$$

(b) The simply supported boundary condition

$$u = 0 \text{ on } \partial\Omega.$$

We partition the domain  $\Omega$  into the uniform squares with the meshsize  $h = \frac{1}{N}$  for some integer. The first six eigenvalues are listed in Table 1, Table 2, respectively.

In the second example, we consider the following eigenvalue problem of the three-dimensional biharmonic equation imposed the following boundary conditions

$$\Delta^2 u = \lambda u \text{ in } \Omega = [0, 1]^3, \quad (4.2)$$

(c) The clamped boundary condition

$$u = \partial_\nu u = 0 \text{ on } \partial\Omega.$$

(d) The simply supported boundary condition

$$u = 0 \text{ on } \partial\Omega.$$

We partition the domain  $\Omega$  into the uniform cubics with the meshsize  $h = \frac{1}{N}$  for some integer. The first six eigenvalues are listed in Table 3, Table 4, respectively.

From the tables, we can find that the discrete eigenvalues converge monotonically from below to the exact ones for all the boundary conditions under consideration.

**Remark 4.1.** *In this paper, we provide the proof of lower bounds of eigenvalues for the clamped boundary condition. However, the analysis in this paper does not cover the case for the simply supported boundary condition. From the numerical results, we can find that it also holds for the simply supported boundary condition.*

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Table 1: The first six eigenvalues for the clamped boundary condition in 2-D case

N	4	8	12	16	32	
$\lambda_{1,h}$	1075.8563	1223.1076	1261.1771	1275.5592	1289.9935	$\nearrow$
$\lambda_{2,h}$	4481.4554	5017.6904	5205.0626	5280.6461	5359.1648	$\nearrow$
$\lambda_{3,h}$	4481.4554	5017.6904	5205.0626	5280.6461	5359.1648	$\nearrow$
$\lambda_{4,h}$	7697.5590	9953.5911	10819.5084	11183.7787	11572.2467	$\nearrow$
$\lambda_{5,h}$	15704.3199	16244.1142	16743.9469	16971.6555	17222.4239	$\nearrow$
$\lambda_{6,h}$	16296.5202	16520.5023	16955.9294	17162.3431	17393.3846	$\nearrow$

Table 2: The first six eigenvalues for the simply supported boundary condition in 2-D case

N	4	8	12	16	32		Exact
$\lambda_{1,h}$	347.5266	377.6791	384.1862	386.5430	388.8563	$\nearrow$	$4\pi^4 \approx 389.6364$
$r$	—	1.816267	1.937758	1.968793	1.987512		
$\lambda_{2,h}$	2104.3141	2323.3219	2382.3420	2404.8176	2427.4598	$\nearrow$	$25\pi^4 \approx 2435.2273$
$\lambda_{3,h}$	2104.3141	2323.3219	2382.3420	2404.8176	2427.4598	$\nearrow$	$25\pi^4 \approx 2435.2273$
$r$	—	1.564173	1.848565	1.923527	1.969013		
$\lambda_{4,h}$	4428.5078	5560.4260	5905.5665	6042.8650	6184.6886	$\nearrow$	$64\pi^4 \approx 6234.1818$
$r$	—	1.422239	1.770756	1.880399	1.950661		
$\lambda_{5,h}$	8883.3154	9298.3330	9516.2149	9608.4258	9706.2378	$\nearrow$	$100\pi^4 \approx 9740.9091$
$\lambda_{6,h}$	8883.3154	9298.3330	9516.2149	9608.4258	9706.2378	$\nearrow$	$100\pi^4 \approx 9740.9091$
$r$	—	0.954369	1.671838	1.836346	1.933997		

Table 3: The first six eigenvalues for the clamped boundary condition in 3-D case

N	4	8	12	16	
$\lambda_{1,h}$	1714.3524	2136.8429	2255.9156	2302.1447	$\nearrow$
$\lambda_{2,h}$	5174.6283	6369.4367	6796.5628	6972.4742	$\nearrow$
$\lambda_{3,h}$	5174.6283	6369.4367	6796.5628	6972.4742	$\nearrow$
$\lambda_{4,h}$	5174.6283	6369.4367	6796.5628	6972.4742	$\nearrow$
$\lambda_{5,h}$	8539.6777	11655.1631	12920.9204	13468.3120	$\nearrow$
$\lambda_{6,h}$	8539.6777	11655.1631	12920.9204	13468.3120	$\nearrow$

Table 4: The first six eigenvalues for the simply supported boundary condition in 3-D case

N	4	8	12	16		Exact
$\lambda_{1,h}$	718.3621	828.0498	854.1259	863.7983	$\nearrow$	$9\pi^4 \approx 876.6818$
$r$	—	1.702863	1.894823	1.946762		
$\lambda_{2,h}$	2720.0885	3226.6792	3372.2667	3428.9320	$\nearrow$	$36\pi^4 \approx 3506.7273$
$\lambda_{3,h}$	2720.0885	3226.6792	3372.2667	3428.9320	$\nearrow$	$36\pi^4 \approx 3506.7273$
$\lambda_{4,h}$	2720.0885	3226.6792	3372.2667	3428.9320	$\nearrow$	$36\pi^4 \approx 3506.7273$
$r$	—	1.490027	1.809503	1.902066		
$\lambda_{5,h}$	5246.9541	6842.3245	7369.5014	7584.7868	$\nearrow$	$81\pi^4 \approx 7890.1364$
$\lambda_{6,h}$	5246.9541	6842.3245	7369.5014	7584.7868	$\nearrow$	$81\pi^4 \approx 7890.1364$
$r$	—	1.334896	1.724958	1.854797		